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UNSTABLE BORDISM GROUPS AND ISOLATED SINGULARITIES

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ABSTRACT. An isolated singularity of an embedded submanifold can be topologically smoothed if and only if a certain obstruction element in $\pi_*(MG)$ vanishes, where G is the group of the normal bundle. In fact this obstruction lies in a certain subgroup which is referred to here as the unstable G-bordism group. In this paper some of the unstable SO-bordism groups are computed; the obstruction to smoothing the complex cone on an oriented submanifold $X \subset \mathbf{C}P^n$ at ∞ is computed in terms of the characteristic numbers of X. Examples of nonsmoothable complex cone singularities are given using these computations.

0. Introduction. In this paper the notion of unstable bordism is introduced and defined in such a way that the obstruction to smoothing an isolated singularity is an element of an unstable bordism group.

Suppose $p \in V$ is an isolated singularity of an otherwise smoothly embedded manifold $V \subset W$ (that is $V \setminus \{p\} \subset W$ is a smooth embedding). The link L of p is by definition the transverse intersection $\partial B \cap (V \setminus \{p\})$ where B is a small ball about p in W. We say that the singularity at p is "smoothable" if L bounds some manifold which is contained in B. Since L is always the boundary of the manifold $V \setminus \{\text{interior of } B\}$ the smoothability of p is not simply a question of whether L represents zero in the appropriate bordism group. The question of smoothability is unstable in the sense that it depends upon the codimension of L in ∂B. Depending on the structure we require on the normal bundle of the link L in ∂B we obtain the notions of G-smoothability and unstable G-bordism where $G = 0$, SO, U, Sp, etc.

In this paper we restrict our attention to the case $G = SO$. We compute some of the unstable oriented bordism groups and then calculate the obstruction element for a certain class of isolated singularities. More precisely, if $X \xrightarrow{b} \mathbf{C}P(k)$ is an embedding of a real oriented k dimensional manifold in complex projective k space, we calculate the obstruction to smoothing the singularity of the complex cone $CX \subset \mathbf{C}P(k+1)$ in terms of characteristic numbers of b. The results are applied to give examples of nonsmoothable complex cone singularities and to extend results obtained by E. Rees and E. Thomas [**8**] for the case $G = U$.

1. Preliminaries. In this section we make precise the notions of smoothing isolated singularities and of unstable oriented bordism groups. We show that the obstruction to smoothing an isolated singularity of an oriented manifold is an element of an unstable oriented bordism group.

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Throughout this paper all manifolds are assumed to be smooth and oriented.

1.1. DEFINITION. Let Y be a manifold of dimension $> k$. A *codimension-k oriented isolated singularity* in Y is a triple (X, p, f) satisfying:

(i) $p \in X$,

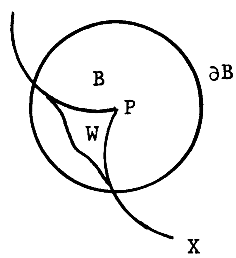
(ii) $X \setminus \{p\}$ is a smooth oriented manifold,

(iii) f: $X \to Y$ is continuous,

(iv) $f|_{X \setminus \{p\}}$: $X \setminus \{p\} \to Y$ is a smooth orientation preserving embedding of codimension-k.

If the point p is understood we write (X, f).

We say that (X, p, f) is *orientably smoothable* if for any sufficiently small ball B about $f(p)$ in Y which is transverse to $X \setminus \{p\}$, there exist a submanifold $W \subset B$ and a bundle isomorphism $(\partial B, \partial W) \to (\partial B, (X \setminus \{p\}) \cap \partial B)$. In particular $\partial W \cong (X \setminus \{p\}) \cap \partial B$. Here is a representation for $k = 1$:



1.2. DEFINITION. Let Ω_* denote the oriented bordism ring; if X is a manifold we denote by $[X]$ the element of Ω_* represented by X.

Let $k < n$ be positive integers and set $\mathfrak{L}_k^n = \{(L, \alpha)/\alpha: L \to S^n$ is a codimension-k orientation preserving embedding and $[L] = 0 \in \Omega_*\}$. We say that (L_1, α_1) and (L_2, α_2) are *ambiently bordant*, written $(L_1, \alpha_1) \sim (L_2, \alpha_2)$ if there exist a manifold W with $\partial W \cong L_1 + (-L_2)$, and α: $(W, \partial W) \to (S^n \times I, S^n \times 0 \cup S^n \times 1)$ such that $\alpha|_{L_1} = \alpha_1$ and $\alpha|_{L_2} = \alpha_2$. It is not hard to check that $\sim$ is an equivalence relation; we define the *nth unstable bordism group of codimension-k oriented manifolds* denoted Λ_k^n, by $\Lambda_k^n = \mathfrak{L}_k^n/\sim$.

1.3. REMARK. Let Y be a manifold of dimension $n + 1$, and let (X, p, f) be a codimension-k isolated singularity in Y. Let B be a small ball about $f(p)$ in Y, set $S^n = \partial B$. Make the inclusion map j: $S^n \to Y$ transverse to the submanifold $f(X \setminus \{p\}) \subset Y$. Then $j^{-1}(f(X \setminus \{p\})) = L$ is a submanifold of S^n of codimension-k which we call the *link* of (X, p, f). The equivalence class of L is an element $\mathfrak{L}(X, p, f) \in \Lambda_k^n$. Choosing a smaller ball $B' \subset B$ or making j transverse in a different manner changes L to an L' which is ambiently bordant to L, hence $\mathfrak{L}(X, p, f)$ depends only on (X, p, f).

1.4. LEMMA. *Let (X, p, f) be a codimension-k isolated singularity in an $n + 1$ manifold. Then (X, p, f) is orientably smoothable if and only if $\mathfrak{L}(X, p, f) = 0 \in \Lambda_k^n$.*

PROOF. $\mathfrak{L}(X, p, f) = 0 \Leftrightarrow$ the embedding $L \subset S^n$ of the link of (X, p, f) extends to an embedding of some W in $S^n \times I$ with $\partial W = L$. This is equivalent to the statement that L bounds in the ball, i.e. (X, p, f) is orientably smoothable.

2. Unstable oriented bordism groups.

2.1. THEOREM. $\Lambda_k^n = 0$ *if* $n \leqslant 2k$,

$$\Lambda_k^{2k+1} = \begin{cases} 0 & \text{if } k+1 \text{ or } k+2 \text{ is a power of } 2, \\ \mathbf{Z}_2 & \text{otherwise.} \end{cases}$$

2.2. THEOREM. *Suppose k is not divisible by* 4. *Then*

$$\Lambda_k^{2k+2} = \begin{cases} 0 & \text{if } k+2 \text{ is a power of } 2, \\ Z_4 & \text{if } k \equiv 1(4), \text{ and} \\ \mathbf{Z}_2 & \text{otherwise.} \end{cases}$$

Moreover if k is odd Λ_k^{2k+3} has a $\mathbf{Z}$ summand.

In each of the above theorems, when the group $\Lambda_k^n = 0$ we can immediately deduce from 1.4 a corollary about smoothing isolated singularities; namely that all codimension-k isolated singularities in $n + 1$ manifolds are orientably smoothable.

The proofs of 2.1 and 2.2 are rather involved; we give an outline of the methods used over the couse of this and the next two sections. First note that the fact that $\Lambda_k^n = 0$ if $n \leqslant 2k - 1$ is merely a reinterpretation of well-known results of R. Thom on oriented bordism groups; this will become more clear as we go along. The reinterpretation of Λ_k^n in terms of the homotopy groups of Thom spaces requires some notation.

2.3. *Notation*. For $k \geqslant 1$ let $M_k = MSO(k)$ denote the Thom space of the universal oriented k-plane bundle γ_k over the classifying space $BSO(k)$. Let b_k: $\Sigma M_k \to M_{k+1}$ be the map from the reduced suspension of M_k to M_{k+1} induced by adding a trivial line bundle to γ_k. The $\{M_k, b_k\}$ forms a spectrum whose homotopy or coefficient ring is the oriented bordism ring Ω_*, that is, $\varinjlim_n \pi_{n+k}(M_n) \cong \Omega_k$, in fact for N sufficiently large Thom has shown that $\pi_{N+k}(M_N) \cong \Omega_k$.

Now for each k let a_k: $M_k \to \Omega M_{k+1}$ denote the adjoint map to b_k; we obtain by composition a map

$$M_k \xrightarrow{a_k} \Omega M_{k+1} \xrightarrow{\Omega a_{k+1}} \Omega^2 M_{k+2} \to \cdots \to \Omega^N M_{k+N} = \Omega(k)$$

which we denote by a: $M_k \to \Omega(k)$. Note that $\pi_m(\Omega(k)) \cong \pi_{m+N}(M_{k+N}) \cong \Omega_{m-k}$. By standard arguments we may assume that the map "a" is a fibration; we use F_k to denote its fiber. We then have the fibration

$$F_k \xrightarrow{i} M_k \xrightarrow{a} \Omega(k). \tag{1}$$

2.4. PROPOSITION. $\Lambda_k^n \cong \text{Kernel}\{a_*\colon \pi_n(MSO(k)) \to \pi_n\Omega(k)\}$.

PROOF. The Thom-Pontryagin construction gives a 1-1 correspondence between the two sets. We define the group structure on Λ_k^n so that it gives an isomorphism of groups.

In view of 2.4 and the homotopy exact sequence of the fibration 2.3(1) above, $\Lambda_k^n \cong \pi_n(F_k)/\partial(\pi_{n+1}(\Omega(k)))$ where ∂: $\pi_{n+1}(\Omega(k)) \cong \Omega_{n+1-k} \to \pi_n(F_k)$ is the boundary map in the homotopy sequence. Thus in order to compute Λ_k^n we need to compute $\pi_n(F_k)$ and evaluate the map ∂.

Our method for computing $\pi_n(F_k)$ in the range of dimensions of interest here involves computing the cohomology of F_k and then using Postnikov type arguments to deduce its homotopy. The cohomology of F_k is computed using the Serre exact squence of fibration 2.3(1) together with knowledge of $H^*(M_k)$ and $H^*(\Omega(k))$.

Now $H^*(M_k)$ is well known, $\Omega(k)$ however is an iterated loop space, namely $\Omega^N M_{k+N}$, so the next section is a digression which gives a method for determining the cohomology of iterated loop spaces. For an alternate method see Milgram [7]. In §4 we complete the outline of the proofs of 2.1 and 2.2.

3. A spectral sequence for the cohomology of iterated loop spaces. We give a method for obtaining information about the cohomology of the Nth iterated loop space $\Omega^N X$ of a space X. More precisely we exhibit a spectral sequence which converges to a natural filtration of $H^*(\Omega^N X)$. This spectral sequence is valid in a "metastable" range of dimensions depending on the connectivity of $\Omega^N X$. In the case of field coefficients the E_2-term is computed enabling in particular a description of $H^*(\Omega^N X; \mathbf{Z}_2)$ in terms of the action of the Steenrod squares on $H^*(X)$.

During the subsequent discussion X denotes an $N + k - 1$ connected space where $N > k > 1$ are fixed.

Results stated without specific mention of a coefficient ring are valid for arbitrary coefficients. The letter Λ will be used for a field of characteristic $\neq 2$.

σ: $H^{r+1}(Y) \to H^r(\Omega Y)$ denotes the "loop map" in cohomology and Δ: $Y \to Y \wedge Y$ denotes the map induced by the diagonal map $Y \to Y \times Y$. Here $Y \wedge Y = Y \times Y / Y \vee Y$ is the usual smash product.

3.1. THEOREM (G. W. WHITEHEAD). *Let Y be n-connected, $n \geqslant 1$. Then for $r \leqslant 3n$ there exists a homomorphism δ: $H^{r-1}(\Omega Y) \to H^{r+1}(Y \wedge Y)$ such that the following sequence is exact*:

$$\cdots H^{r-1}(\Omega Y) \xrightarrow{\delta} H^{r+1}(Y \wedge Y) \xrightarrow{\Delta} H^{r+1}(Y) \xrightarrow{\sigma} H^r(\Omega Y) \to \cdots \to H^{3n}(Y).$$

Moreover if $u \in H^i(Y)$ and $v \in H^j(Y)$ then

$$(*) \qquad \delta(\sigma u \cdot \sigma v) = (-1)^i \left[u \wedge v - (-1)^{ij} v \wedge u \right].$$

PROOF. The existence of a "δ" such that the sequence is exact is proved in Whitehead [**9**]. The sequence above as obtained by Whitehead is natural so it is enough to show that δ can be chosen satisfying $(*)$ when Y is a product of Eilenberg-Mac Lane spaces. This is fairly straighforward and is left to the reader.

3.2. COROLLARY. *Let X be $(N + k - 1)$-connected. Then*

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & D \\ & {\scriptstyle\gamma}\nwarrow \quad \swarrow{\scriptstyle\beta} & \\ & E & \end{array}$$

is an exact couple where D, E, α, β, γ are defined as follows:

$$D^{s,t} = \begin{cases} H^s(\Omega^t X) & \text{if } 0 \leq t \leq N, s \leq 3(N+k-t)-1, \\ 0 & \text{if } t \leq 0, \\ D^{s+1,t-1} & \text{if } t \geq N+1 \text{ or } s > 3(N+k-t)-1; \end{cases}$$

$$E^{s,t} = \begin{cases} H^{s+2}(X) & \text{if } t = -1, s \leq 3(N+k-1), \\ H^s(\Omega^t X \wedge \Omega^t X) & \text{if } 0 \leq t \leq N-1, s \leq 3(N+k-t-1), \\ \delta H^{s-2}(\Omega^{t+1} X) & \text{if } 0 \leq t \leq N-1, s \leq 3(N+k-t-1)+1, \\ 0 & \text{otherwise}. \end{cases}$$

α: $D^{s,t} \to D^{s-1,t+1}$ is σ: $H^s(\Omega^t X) \to H^{s-1}(\Omega^{t+1}X)$ for $s \leq 3(N+k-t-1)$ and is the identity otherwise. γ: $E^{s,t} \to D^{s,t}$ is Δ^*: $H^s(\Omega^t X \wedge \Omega^t X) \to H^s(\Omega^t X)$ for $s < 3(N+k-t)-1$ and is 0 otherwise. β: $D^{s-2,t+1} \to E^{s,t}$ is the identity if $t = -1$ is the zero map if $t < -1$ or $t > N-1$ or $s > 3(N+k-t)-1$, and is the map δ: $H^{s-2}(\Omega^{t+1}X) \to H^s(\Omega^t X \wedge \Omega^t X)$ otherwise.

The above corollary is proved by applying the Whitehead theorem repeatedly for $Y = \Omega^t X$, $t = 1, 2, \ldots$. This exact couple gives rise as usual to a spectral sequence:

3.3. THEOREM. *Let X be an $(N+k-1)$-connected space with $N, k \geq 1$. Then there is a spectral sequence $\{E_r^{s,t}, d_r\}$ such that*

(i) *d_r has bidegree $(r+1, -r)$,*

(ii)

$$E_1^{s,t} \cong \begin{cases} H^s(\Omega^t X \wedge \Omega^t X) & \text{if } s \leq t \leq N-1, s \leq 3(N+k-t-1), \\ H^s(X) & \text{if } t = -1, \\ 0 & \text{otherwise}, \end{cases}$$

(iii) *$E_\infty^{r,N-s} \cong F^{r-2,s-1}/F^{r-1,s}$ if $r \leq 3(k+s-1)$.*
Here $F^{r,s} = \sigma^s H^r(\Omega^{N-s}X)$.

The above theorem gives for $n \leq 3(k-1)$ a filtration:

$$\sigma^N H^{n+N}(X) = F^{n+N,N} \subset \cdots \subset F^{n+1,1} \subset F^{n,0} = H^n(\Omega^N X)$$

to which the spectral sequence converges. Hence we obtain information about $H^*(\Omega^N X)$ in terms of $H^*(\Omega X \wedge \Omega X)$.

We now turn our attention to obtaining more specific results when the coefficient ring is a field. In this case

$$E_1^{s,t} \cong H^s(\Omega^t X \wedge \Omega^t X) \cong \sum_{p+q=s+2t} H^p(X) \otimes H^q(X)$$
$$\cong H^{s+2}(\Omega^{t-1} X \wedge \Omega^{t-1} X) \cong E_1^{s+2,t-1},$$

so the first differential can be identified with an automophism of the vector space $\sum H^p(X) \otimes H^q(X)$. Property $(*)$ of the map δ (see 3.1) can then be used to compute d_1. A close examination of the exact couple lattice then allows all the differentials to be computed. We obtain the following results when Λ is a field of characteristic $\neq 2$:

3.4. THEOREM. *Let X be $(N+k-1)$-connected, $N > k+1 \geqslant 2$. Suppose $H^q(X;\Lambda)$ has as basis over Λ the set $\{u_{q,1},\dots,u_{q,n(q)}\}$ for each q. Let $s \leqslant 3(k-1)$. Then $H^s(\Omega^N X,\Lambda)/\sigma^N H^{N+s}(X;\Lambda)$ has as basis:*

$$\left\{\left(\sigma^N u_{N+p,i}\right)\cdot\left(\sigma^N u_{N+q,j}\right)/p+q=s\right\} \quad \textit{if } s \equiv 0 \pmod 4,$$

$$\left\{\left(\sigma^N u_{N+p,i}\right)\cdot\left(\sigma^N u_{N+q,j}\right)/p+q=s \textit{ and } p=q \Rightarrow i \neq j\right\} \quad \textit{otherwise}.$$

3.5. THEOREM. *Let X be $(N+k-1)$-connected, $s \leqslant 3(k-1)$, $N > k+1$, $1 \leqslant t \leqslant N$. Then*

$$\operatorname{Ker}\{\sigma^t\colon H^{s+t}(X;\Lambda) \to H^s(\Omega^t X;\Lambda)\} = 0.$$

The results corresponding to the above theorems in the case of $\mathbf{Z}_2$ coefficients are obtained in a similar manner but are made more complicated by the involvement of the Steenrod squares. To give an idea how they arise we give a more explicit proof of the "$\mathbf{Z}_2$ version" of 3.5, leaving the reader to fill in the details of the $\mathbf{Z}_2$ version of 3.4. First we need the following definition.

3.6. DEFINITION. We say that an element $w \in H^s(X,\mathbf{Z}_2)$ is *Steenrod-decomposable* of degree (s,t) if either

(i) s is even and w is in the span of

$$\{\Delta^*(H^s(X\wedge X;\mathbf{Z}_2)) \cup Sq^{s/2-m}H^{s/2+m}(X;\mathbf{Z}_2)/1 \leqslant m \leqslant t\},$$

or

(ii) s is odd and $w \in \operatorname{span}\{\Delta^* H^s(X\wedge X;\mathbf{Z}_2)\}$.

We denote by $\mathfrak{D}^{s,t}$ the set of all S-decomposable elements of degree (s,t). Note that "t" need not be an integer.

3.7. THEOREM. *Let X be $(N+k-1)$-connected, $1 \leqslant t \leqslant N$, $s-t \leqslant 3(N+k-t-1)$. Then the kernel of the map $\sigma^t\colon H^s(X,\mathbf{Z}_2) \to H^{s-t}(\Omega^t X;\mathbf{Z}_2)$ is $\mathfrak{D}^{s,(t-1)/2}$.*

PROOF. We proceed by induction on t; for $t=1$ the result follows from the Whitehead exact sequence (see 3.1). We assume the result for positive integers less than t, and examine a portion of the exact couple lattice:

$$\begin{array}{c} H^s(X) \\ \sigma\downarrow \\ H^{s-1}(\Omega X) \\ \downarrow \\ \vdots \\ \downarrow \end{array}$$

$$H^{s-t+1}(\Omega^{t-1}X\wedge\Omega^{t-1}X) \xrightarrow{\Delta^*} H^{s-t+1}(\Omega^{t-1}X) \xrightarrow{\delta} H^{s-t+3}(\Omega^{t-2}X\wedge\Omega^{t-2}X).$$

Suppose $x \in H^s(X)$ and $\sigma^{t-1}x \neq 0$ but $\sigma^t x = 0$. Then $\sigma^{t-1}x = \Delta^*(\sigma^{t-1}u \wedge \sigma^{t-1}v)$ for some $u, v \in H^*(X)$ (exactness in 3.1). But

$$0 = \delta(\sigma^{t-1}x) = \delta((\sigma^{t-1}u)\cdot(\sigma^{t-1}v)) = \sigma^{t-2}u \wedge \sigma^{t-2}v + \sigma^{t-2}v \wedge \sigma^{t-2}u$$

(by 3.1(*)); this happens only when $u = v$, hence $\sigma^{t-1}x = (\sigma^{t-1}u)^2 = Sq^{s/2-t+1}u$. Therefore $x + Sq^{s/2-t+1}u \in \ker \sigma^{t-1} = \mathfrak{D}^{s,(t-2)/2}$ (by induction), which implies $x \in \mathfrak{D}^{s,(t-1)/2}$.

3.8. THEOREM. *Let X be $(N + k - 1)$-connected, $N > k + 1$, $s \leqslant 3(k - 1)$. Suppose $\{u_j/j \in J\}$ is a basis for $H^*(X; \mathbf{Z}_2)$, and set $A^{p,q} = \{u_j/Sq^{p/2}u_j \in \mathfrak{D}^{p+q,(q-1)/2}\}$. For each $u_j \in A^{p,q}$ pick $\hat{u}_j$ with $\delta u_j = \sigma^q u_j$ and let $B^{p,q} = \{\hat{u}_j/u_j \in A^{p,q}\}$. Then*

$$\cup \{B^{p,q}/p + q = N + n + 1, 0 \leqslant q \leqslant N - 1\}$$

$$\cup \left\{(\sigma^N u_i) \cdot (\sigma^N u_j)/u_i \in H^{l+n}(X), u_j \in H^{m+N}_{l \leqslant m, i \neq j}(X), l + m = s\right\}$$

is a basis for $H^s(\Omega^N X; \mathbf{Z}_2)/\sigma^N H^{N+s}(X; \mathbf{Z}_2)$.

4. Proofs of Theorems 2.1 and 2.2.

4.0. *Notation*. The Thom class in $H^{N+k}(MSO(k + N)) = H^{N+k}(M_{k+N})$ is denoted U_{k+N} or U. For a finitely nonzero sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ we have monomials $w_\alpha = w_1^{\alpha_1} w_2^{\alpha_2} \cdots w_r^{\alpha_r}$ and $p = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ in the Stiefel-Whitney and Pontryagin classes respectively. We denote their images under the Thom isomorphism by $w_\alpha U$ and $p_\alpha U$ respectively. If $|\alpha| = \alpha_1 + 2\alpha_2 + \cdots + r\alpha_r$ then $w_\alpha U \in H^{N+k+|\alpha|}(M_{k+N}; \mathbf{Z}_2)$ and $p_\alpha U \in H^{N+k+4|\alpha|}(M_{k+N}; \Lambda)$ (Λ as in §3 is a field of characteristic $\neq 2$, so we identify $p_\alpha U$ as the image under the coefficient homomorphism $\mathbf{Z} \to \Lambda$, of the Pontryagin classes).

We use the notation $w_\alpha V$, $p_\alpha V$ for the images $\sigma^N(w_\alpha U)$, $\sigma^N(p_\alpha U)$ of the above classes under the N-fold iteration of the loop map.

The results of the previous section applied to $X = \Omega^N M_{k+N}$ yield the following.

4.1. THEOREM. *Suppose k is odd and $s \leqslant 3(k - 1)$. Then*

$$H^s(\Omega(k); \Lambda)/\sigma^N H^{s+N}(M_{k+N}; \Lambda) = 0$$

unless $s = 2k + 4n$ for some $n \geqslant 1$; in which case it has as basis (over Λ)

$$\left\{(p_\alpha V) \cdot (p_\beta V)/|\alpha| + |\beta| = n, |\alpha| \leqslant |\beta|, \alpha \neq \beta\right\}.$$

4.2. THEOREM. *Suppose k is even and $s \leqslant 3(k - 1)$. Then*

$$H^s(\Omega(k); \Lambda)/\sigma^N H^{s+N}(M_{k+N}; \Lambda) = 0$$

unless $s = 2k + 4n$ for some $n \geqslant 1$ in which case it has as basis $\{(p_\alpha V) \cdot (p_\beta V)/|\alpha| + |\beta| = n, |\alpha| \leqslant |\beta|\}$.

4.3. THEOREM. *Let Λ be a field of characteristic $\neq 2$, $s \leqslant 3(k - 1)$. Then σ^N: $H^{s+N}(M_{k+N}; \Lambda) \to H^s(\Omega(k); \Lambda)$ is a monomorphism.*

For $\mathbf{Z}_2$ coefficients we have

4.4. THEOREM. *If $s \leqslant 3(k - 1)$ then*

$$\left\{(w_\alpha V)(w_\beta V)/|\alpha| + |\beta| = s - 2k, |\alpha| \leqslant |\beta|, \alpha \neq \beta\right\}$$

is a basis for the $\mathbf{Z}_2$ vector space

$$H^s(\Omega(k); \mathbf{Z}_2)/\sigma^N H^{s+N}(M_{k+N}; \mathbf{Z}_2).$$

4.5. THEOREM. *Let $s \leqslant 3(k-1)$. Then the kernel of σ^N: $H^{s+N}(M_{k+N};\mathbf{Z}_2) \to H^s(\Omega(k);\mathbf{Z}_2)$ consists of exactly those elements which are Steenrod-decomposable of degree $(s,(N-1)/2)$ (see 3.6).*

Deducing the cohomology of F_k from the above theorems is a matter of examining the Serre exact sequence of the fibration $F_k \to M_k \to \Omega(k)$ together with the following commutative diagram:

$$\begin{array}{ccc} H^{l+N}(M_{k+N}) & \xrightarrow{b^*} & H^{l+N}(\Sigma^N M_k) \\ \sigma^N \downarrow & & \cong\uparrow \Sigma^N \\ H^l(\Omega(K)) & \xrightarrow{a^*} & H^l(M_k) \end{array}$$

Here b is obtained by composing the structure maps of the Thom spectrum, $\Sigma^N M_k \to \Sigma^{N-1}M_{k+1} \to \cdots \to M_{k+N-1} \to M_{k+N}$, and hence is adjoint to a. Σ^N, σ^N denote the N-fold iteration of the suspension and loop maps respectively.

Once $H^*(F_k;\mathbf{Z}_2)$ and $H^*(F_k;\Lambda)$ have been deduced as indicated above one obtains $H^*(F_k;\mathbf{Z})$ in dimensions close to $2k$ using knowledge of the action of the Steenrod squares in $H^*(F_k;\mathbf{Z}_2)$ and by applying the Bockstein spectral sequence for F_k. The action of the Steenrod squares is computed by noting their action on $H^*(MSO(k);\mathbf{Z}_2)$ together with the fact that Sq^i commutes with σ, τ, Σ. The key differentials of the Bockstein sequence can be computed using results of Browder [**2**]. We obtain

4.6. THEOREM. *Let $k \geqslant 5$. Then $H^l(F_k;\mathbf{Z}) = 0$ for $l \leqslant 2k+1$, $H^{2k+2}(F_k;\mathbf{Z}) = \mathbf{Z}_2$.*

$$H^{2k+3}(F_k;\mathbf{Z}) \cong \begin{cases} \mathbf{Z}_2 & \text{if } k \text{ is even}, \\ \mathbf{Z}_2 \oplus \mathbf{Z} & \text{if } k \text{ is odd}; \end{cases}$$

$$H^{2k+4}(F_k;\mathbf{Z}) \cong \begin{cases} \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_4 & \text{if } k \text{ is even}, \\ \mathbf{Z}_2 \oplus \mathbf{Z}_2 & \text{if } k \text{ is odd}. \end{cases}$$

Postnikov type arguments now yield

4.7. THEOREM. *Let F_k be the fiber of a: $MSO(k) \to \Omega(k)$ and suppose $k \geqslant 5$. Then*

	$k \equiv 0(4)$	$k \equiv 1(4)$	$k \equiv 2(4)$	$k \equiv 3(4)$
$\pi_{2k+1}(F_k) \cong$	$\mathbf{Z}_2$,	$\mathbf{Z}_2$,	$\mathbf{Z}_2$,	$\mathbf{Z}_2$
$\pi_{2k+2}(F_k) \cong$	$\mathbf{Z}_2 \oplus \mathbf{Z}_2$,	$\mathbf{Z}_4$,	$\mathbf{Z}_2$,	$\mathbf{Z}_2$
$\pi_{2k+3}(F_k) \cong$	$\mathbf{Z}_2 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_4$,	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}$,	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_4$,	$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}$

The methods outlined above actually give more than simply the groups in 4.6 and 4.7, they also allow determination of the generators of those groups in most cases. For instance a map φ: $S^{2k+1} \to F_k$ is essential if and only if $\varphi^*(f) \neq 0 \in H^{2k+1}(S^{2k+1};\mathbf{Z}_2)$ where $f \in H^{2k+1}(F_k;\mathbf{Z}_2)$ is determined by the fact that $\tau f = Vw_2V + w_2w_kV \in H^{2k+2}(\Omega(k);\mathbf{Z}_2)$. ($\tau$ denotes transgression.) This explicit knowledge of the generators of $\pi_*(F_k)$ allows us to calculate the boundary map ∂: $\Omega_* \to \pi_*(F_k)$ and hence the groups Λ_k^n for n close to $2k$. We use the following lemma concerning the boundary map ∂ in the homotopy sequence of a fibration.

4.8. LEMMA. *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a fibration and suppose g: $S^{n+1} \to B$ with $\partial\{g\} = \{h\} \in \pi_n(F)$. Then $\langle \tau x, g_*[S^{n+1}]\rangle = \langle x, h_*[S^n]\rangle$ for any $x \in H^n(F)$.*

Here $[S^{n+1}]$, $[S^n]$ are the canonical generators in cohomology and $\langle\ ,\ \rangle$ denotes the Kronecker pairing.

Applying 4.8 to the fibration $F_k \to M_k \to \Omega(k)$ we obtain results on its boundary map; in particular in the first case where $\pi_*(F_k) \neq 0$ we have

4.9. PROPOSITION. *The boundary map ∂: $\Omega_{k+2} = \pi_{2k+2}(\Omega(k)) \to \pi_{2k+1}(F_k) \cong \mathbf{Z}_2$ is given by $\partial[X] = w_2w_k[X]$ for any $[X] \in \Omega_{k+2}$. Here $w_2w_k[X] = \langle w_2w_k(\nu), [X]\rangle$ is the normal Stiefel-Whitney number of X ($\nu(X)$ is the normal bundle of X).*

The group $\Lambda_k^{2k+1} \cong \pi_{2k+1}(F_k)/\operatorname{Im}\partial$ can now be deduced from the following result of Massey and Peterson (see **[6]**).

4.10. LEMMA (MASSEY AND PETERSON). *Let M^n be a compact connected oriented manifold with $n > 2$ such that the normal Stiefel-Whitney class $w_{n-2} \neq 0$. Then $n = 2^r + 1$, $r > 0$ or $n = 2^{r+1}$, $r > 0$.*

Since examples are also given in **[6]** where $w_2w_{n-2}[M^n] \neq 0$ when $n = 2^r + 1$ and $n = 2^{r+1}$ we can combine 4.9 and 4.10 to obtain Theorem 2.1. Theorem 2.2 can also be obtained as outlined here.

5. Complex cone singularities.

5.1. DEFINITION. Let $P_n = \mathbf{C}P(n)$ denote complex projective n-space (real dimension $2n$).

Suppose f: $X \to P_n$ is an embedding of a manifold X of real codimension k. We form the complex cone on X (complex lines through X to ∞) in P_{n+1}, obtaining Cf: $CX \to P_{n+1}$. Note that (CX, ∞, Cf) is a codimension-k isolated singularity in P_{n+1}.

We say that (CX, ∞, Cf) is the *complex cone singularity* determined by the pair (X, f). We write $l(X, f)$ for the obstruction element $\mathfrak{L}(CX, \infty, Cf)$. Thus $l(X, f) \in \Lambda_k^{2n+1}$ is the obstruction to smoothing CX at ∞ in P_{n+1}.

In this section we calculate the complex cone obstruction when $n = k$, that is when b: $X \to P_k$ is an embedding of a manifold of real dimension k. Since P_k may be taken as the $2k$-skeleton of a $K(\mathbf{Z}, 2)$ (Eilenberg-MacLane space), b corresponds to an element (still denoted b) of $H^2(X; \mathbf{Z})$. We also denote its reduction modulo 2 by $b \in H^2(X; \mathbf{Z}_2)$.

The obstruction $l(X, b) \in \Lambda_k^{2k+1}$, which is $\mathbf{Z}_2$ unless $k + 1$ or $k + 2$ is a power of 2, in which case $\Lambda_k^{2k+1} = 0$ (see 2.1). Thus if $k + 1$ or $k + 2$ is a power of 2 $l(X, b) = 0$, if not we write $l(X, b) = \bar{l}(X, b) \cdot (gen) \in \mathbf{Z}_2$.

We proceed to calculate the mod 2 integer $\bar{l}(X, b)$. In the following theorems $w_2w_mb^q[X]$ and $w_mb^{q+1}[X]$ denote characteristic numbers (mod 2) of the manifold X, with the convention that w_i means $w_i(\nu)$ where ν is the normal bundle of X.

5.2. THEOREM. *Let $k = 2n$ and suppose $k + 2$ is not a power of 2. Let b: $X \to P_k$ be an embedding with* dim $X = k$. *Then the singularity of $CX \subset P_{k+1}$ at ∞ is orientably smoothable if and only if $\bar{l}(X, b) \equiv 0 \pmod 2$ where*

$$\bar{l}(X, b) = \begin{cases} 0 & \text{if } n \text{ is even}, \\ \displaystyle\sum_{q=n-i}^{n-1} \left(w_2 w_{2(n-q-1)} b^q + w_{2(n-q-1)} b^{q+1} \right)[X] & \text{if } n \text{ is odd}. \end{cases}$$

Here $i = 2^s$ is the lowest power of 2 occurring in the dyadic expansion of $n + 1$.

5.3. THEOREM. *Let $k = 2n + 1$ and suppose $k + 1$ is not a power of 2. Let b: $X \to P_k$ be an embedding with* dim $X = k$. *Then the singularity of $CX \subset P_{k+1}$ at ∞ is orientably smoothable if and only if $\bar{l}(X, b) \equiv 0 \pmod 2$ where*

$$\bar{l}(X, b) = \sum_{r=2}^{2^{r-1} \leq n} \sum_{q=n-2^{r-1}}^{2n-2^r} \binom{n - 2^{r-1}}{q + 2^{r-1} - n} \binom{n + 1 + 2^{r-1}}{2^r} w_2 b^q w_{2(n-q)-1}[X].$$

Before indicating the proofs of 5.2 and 5.3 we give some corollaries.

5.4. COROLLARY. *Let b: $X \to P_k$ be an embedding with $k =$* dim X *odd. Suppose that as a cohomology class $b = 0 \in H^2(X; \mathbf{Z}_2)$. Then*

$$\bar{l}(X, b) = \begin{cases} w_2 w_{k-2}[X] & \text{if } k = 2^s + 1 \text{ some } s \geq 2, \\ 0 & \text{otherwise}. \end{cases}$$

5.5. COROLLARY. *Let b: $X \to P_k$ be an embedding with $k =$* dim X *odd. Suppose that $w_3(X) = 0$. Then $CX \subset P_{k+1}$ is orientably smoothable.*

PROOF. Since X is a k dimensional oriented manifold, $Sq^1 v = 0$ for all $v \in H^{k-1}(X, \mathbf{Z}_2)$; also $Sq^1 b = 0 = Sq^1 w_2 (= w_3)$ because b is the reduction of a $\mathbf{Z}$ class. These two facts are enough to show that the characteristic numbers $b^m w_{k-2m}$ and $b^{m-1} w_2 w_{k-2m}$ vanish. The result then follows from 5.3.

5.6. REMARKS. 5.5 shows that $w_3(X) = 0$ is a sufficient condition for CX to be orientably smoothable. This condition is not necessary however for it is easy to find an embedding b: $X \to P_k$ with CX smoothable but $w_3(X) \neq 0$. For example set $X = Y \times S^m$ with $w_3(Y) \neq 0$, $m \geq 2$ and dim $X = k$ odd. If X is embedded homotopically trivially, $\bar{l}(X, b) = w_2 w_{k-2}[X] = 0$ so CX is smoothable but $w_3(X) \neq 0$.

5.7. DEFINITION. We say an embedding b: $X \to P_k$ is *characteristic* if the cohomology class $b \in H^2(X; \mathbf{Z})$ reduces to $w_2(X) \in H^2(X; \mathbf{Z}_2)$.

5.8. COROLLARY. *Let b: $X \to P_k$ be a characteristic embedding with* dim $X = k$. *Then $CX \subset P_{k+1}$ is orientably smoothable at ∞.*

PROOF. If k is odd this follows from 5.5 because $w_3 = Sq^1 w_2 = Sq^1 b = 0$.

If $k = 2n$ we may assume n is odd (see 5.3). But then by Theorem 5.3 $\bar{l}(X, b) = \Sigma(w_2 b^q + b^{q+1}) w_{2(n-q-1)}[X]$. Since $w_2 b^q = w_2^{q+1} = b^{q+1}$, we have $\bar{l}(X, b) \equiv 0$, hence CX is orientably smoothable.

We now give a breif outline of the essential ideas used in the proofs of 5.2 and 5.3. The essential tool is the complex bordism theory MSO^*, in particular $MSO^*(P_k)$. To use it we need the following lemmas.

5.9. LEMMA: *Let* $S^{2k+1} \xrightarrow{\pi} P_k$ *be the usual fibration,* $b\colon X \to P_k$ *an embedding with* $\dim X = k$. *Suppose* $f\colon P_k \to MSO(k)$ *is obtained from* X *by the Thom-Pontryagin construction. Then* $l(X, b) = [f \circ \pi] \in \Lambda_k^{2k+1} \subset \pi_{2k+1}(MSO(k))$.

PROOF. If S^{2k+1} is thought of as the boundary of a small ball about ∞ in P_{k+1} then π identifies complex lines through $\infty \in P_{k+1}$. The map $f \circ \pi$ can then be viewed as being obtained from the link $CX \cap S^{2k+1}$ of ∞ in P_{k+1}, by the Thom-Pontryagin construction. Thus $f \circ \pi$ represents the obstruction $l(X,b)$ to smoothing CX in P_{k+1}.

5.10. LEMMA. *Let* $X \xrightarrow{b} P_k$ *be an embedding with* $\dim X = k$ *and* $f\colon P_k \to MSO(k)$ *the map obtained from X by the Thom-Pontryagin construction. Then there exist maps e and g so that the following commutes*:

$$(1)\qquad \begin{array}{ccccc} S^{2k+1} & \xrightarrow{\pi} & P_k & \xrightarrow{j} & P_{k+1} \\ e\downarrow & & \downarrow f & & \downarrow g \\ F_k & \xrightarrow{i} & M_k & \xrightarrow{a} & \Omega(k) \end{array}$$

Moreover the following commutes for any cohomology theory h^*:

$$(2)\qquad \begin{array}{ccc} h^{*+1}(P_{k+1}) & \xleftarrow{\delta} & h^*(S^{2k+1}) \\ g^*\uparrow & & \uparrow e^* \\ h^{*+1}(\Omega(k)) & \xleftarrow{\tau} & h^*(F_k) \end{array}$$

PROOF. The top row of (1) is a cofibration, the bottom row is a fibration, hence the result follows from the fact that $a \circ f \circ \pi$ is null homotopic (because $[f \circ \pi] = l(X, b) \in \Lambda_k^{2k+1} = \operatorname{Ker} a_*$).

5.11. DEFINITION. Let Y be a space and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ a finitely nonzero sequence; we define a cohomology operation $\Gamma_\alpha\colon MSO^*(Y) \to H^{*+|\alpha|}(Y; \mathbf{Z}_2)$ using the map of spectra induced by $w_\alpha U\colon MSO(n) \to K(\mathbf{Z}_2, n + |\alpha|)$ where $w_\alpha U$ denotes the image of $w_\alpha \in H^{|\alpha|}(BSO(n); \mathbf{Z}_2)$ under the Thom isomorphism.

In particular we have $\Gamma_{(k,2)}$, Γ and Γ_n induced by $w_2 w_k U$, U and $w_n U$ respectively. From the Cartan formula we deduce the following formulae:

$$(*)\qquad \Gamma_n(u \cdot v) = \sum_{p+q=n} \Gamma_p(u) \cdot \Gamma_q(v),$$

$$(**)\qquad \Gamma_{(n,2)}(u \cdot v) = \sum_{p+q=n} \left(\Gamma_{(p,2)}(u)\Gamma_q(v) + \Gamma_p(u)\Gamma_{(q,2)}(v)\right)$$

for $u, v \in MSO^*(Y)$.

5.12. REMARK. Since

$$\begin{aligned} MSO^k(Y) &\cong \lim[\Sigma^n Y, MSO(k+n)] = [\Sigma^N Y, MSO(k+N)] \\ &\cong [Y, \Omega^N MSO(k+N)] = [Y, \Omega(k)], \end{aligned}$$

a map $Y \xrightarrow{u} \Omega(k)$ represents an element of $MSO^k(Y)$. It is not hard to check that $\Gamma_\alpha(u) = u^*(w_\alpha V)$ (where as usual $w_\alpha V = \sigma^N(w_\alpha U_{k+N}) \in H^{k+|\alpha|}(\Omega(k); \mathbf{Z}_2)$).

We apply this to the map $g\colon P_{k+1} \to \Omega(k)$ of 5.10.

5.13. PROPOSITION. *Let* $b\colon X \to P_k$ *be an embedding with* $\dim X = k$ *and suppose* $g\colon P_{k+1} \to \Omega(k)$ *is chosen as in* 5.10(1). *Then*

$$\bar{l}(X, b) \cdot (gen) = \Gamma[g]\Gamma_2[g] + \Gamma_{2,k}[g] \quad in\ H^{2k+2}(P_{k+1}; \mathbf{Z}_2).$$

PROOF. Use the fact that a map $\varphi\colon S^{2k+1} \to F_k$ is essential $\Leftrightarrow \varphi^*(f) \neq 0$ where $\tau f = Vw_2V + w_2w_kV$ together with the commutativity of 5.10(2) to show that $\bar{l}(X, b) \cdot (gen) = g^*(Vw_2V + w_2w_kV)$. The result then follows from the above remarks.

To evaluate Γ, Γ_2, $\Gamma_{(k,2)}$ on $[g]$ we use

5.14. LEMMA. *Let* $y\colon P_k \to MSO(2) = CP(\infty)$ *be the natural map. Then* $MSO^*(P_k) \cong MSO_*[y]/(y^{k+1})$ *where* $MSO_* = \Omega_*$ *is the coefficient ring of MSO.*

PROOF. The map y is an orientation for the cohomology theory MSO^* so the result is standard (see Adams [**1**]).

5.15. LEMMA. *Let* $b\colon X \to P_k$ *with* $\dim X = k$ *and suppose* $f\colon P_k \to MSO(k)$ *is obtained by applying the Thom-Pontryagin construction to X. Then the map* $g\colon P_{k+1} \to \Omega(k)$ *of* 5.10 *can be chosen so that*

$$[g] = \sum_{i=0}^{k} u_i y^{(i+k)/2} \in MSO^k(P_{k+1})$$

where

$$u_i = X \cap P_{(i+k)/2} + \sum_{m=0}^{i-1} u_m P_{(i-m)/2} \in MSO_*.$$

(u_i *is defined inductively on i.*)

Here $P_{q/2}$ *and* $Y^{q/2}$ *are to be interpreted as* 0 *if q is odd.*

5.13 and 5.15 together with 5.11(*), (**) can now be used to obtain the following propositions.

5.16. PROPOSITION. *Let* $X \xrightarrow{b} P_k$ *be an embedding with* $\dim X = k$ *that* $k + 1$ *is not a power of* 2. *Then*

$$\bar{l}(X, b) = \sum_{j=(k+1)/2}^{k} w_2 w_{2j-k-2}[X \cap P_j] \cdot \binom{j}{k-j+1}$$

where $w_2w_{2j-k-2}[X \cap P_j]$ *is a normal S-W number and the binomial coefficients are reduced* mod 2.

5.17. PROPOSITION. *Let* $X \xrightarrow{b} P_k$ *be an embedding with* $\dim X = k = 2n$, *and suppose* $n + 1$ *is not a power of* 2. *Then*

$$\bar{l}(X, b) = \sum_{i=1}^{n} w_{2(i-1)} w_2 [X \cap P_{n+i}] \cdot \binom{n+i}{2i-1}.$$

The total Stiefel-Whitney class $w(X \cap P_j)$ can be computed in terms of $w(X)$ and the embedding b using knowledge of $w(P_j)$. Theorems 5.2 and 5.3 then follow from 5.16 and 5.17.

6. Examples. The results of the previous section give examples and general cases of orientably smoothable complex cone singularities. A natural quesion to ask is whether all complex cone singularities are orientably smoothable. The answer to this question is no. In this section we give some examples of nonsmoothable complex cone singularities.

We first consider the case $k = 2n$. We know that if $n + 1$ is a power of 2, all isolated singularites of codimension k in a $2k + 2$ manifold are smoothable (because $\Lambda_k^{2k+1} = 0$ in this case). Moreover 5.2 says that if n is even all complex cone singularities are orientably smoothable. We exhibit examples of nonsmoothable cone singularities in the other cases, that is when n is odd and $n + 1$ is not a power of 2.

6.1. PROPOSITION. *Let $k = 2n$ with n odd and $n + 1$ not a power of 2. Let $i = 2^s$ be the lowest power of 2 occurring in the dyadic expansion of $n + 1$, and b: $P_i \times P_{n-i} = X \to P_k$ be the algebraic projection of the Segre embedding.*

Then the complex cone $CX \subset P_{k+1}$ is not orientably smoothable.

PROOF. By 5.2,

$$\bar{l}(X, b) = \sum_{q=n-i}^{n-1} \left(w_2 w_{2(n-q-1)} b^q + w_{2(n-q-1)} b^{q+1}\right)[X].$$

b is the Segre embedding so it represents the class $x + y \in H^2(P_i \times P_{n-i}; \mathbf{Z}_2)$, where $x \in H^2(P_i)$ and $y \in H^2(P_{n-i})$ are the usual generators.

Using the fact that

$$\binom{m}{q} \equiv \prod_r \binom{\alpha_r(m)}{\alpha_r(q)} \pmod 2$$

where $m = \sum \alpha_r(m)2^r$, $q = \sum\alpha_r(q)2^r$ are the dyadic expansions of m, q respectively, a fairly straightforward computation yields $\bar{l}(X, b) \equiv 1 \pmod 2$, hence CX is not orientably smoothable.

We now turn to the case where k is odd. Here our information on complex cone singularities is not as complete. We do know (see 5.3) that if $k + 1$ is a power of 2 then all codimension-k isolated singularities in $2k + 2$ manifolds are smoothable. We are also able to obtain examples of nonsmoothable complex cone singularities when $k = 2^s + 1$.

6.2. PROPOSITION. *Let $X = P(1, 2^{s-1})$ be the Dold manifold of dimension $2^r + 1$ (obtained from $S^1 \times P_{2^s-1}$ using the antipodal and conjugate actions). Let b: $X \to P_{2^s+1}$ be an embedding which is homotopic to the constant map.*

Then $CX \subset P_{2^s+2}$ is not orientably smoothable.

Note. An embedding b as above exists by virtue of results of Haefliger [**5**].

PROOF. By 5.4 it suffices to check that $w_2 w_{k-2}[X] \neq 0$ which is well known. (This in fact is the example given in [**6**] of a k-manifold with normal class $w_{k-2} \neq 0$.)

Propositions 6.1 and 6.2 give the generator of $\Lambda_k^{2k+1} \cong \mathbf{Z}_2$ for certain k as obstructions to smoothing complex cone singularities. Thus for these k the generator of Λ_k^{2k+1} is obtained by applying the Thom-Pontryagin construction to the link $L \subset S_k^{2k+1}$ of the appropriate complex cone singularity.

But for $b: X \to P_k$, one obtains using diagram 5.10(1) the pull-back diagram:

$$\begin{array}{ccc} L & \subset & S^{2k+1} \\ \downarrow & & \downarrow \\ X & \xrightarrow{b} & P_k \end{array}$$

thus L is the circle bundle over X induced by the map b. Thus we have given an explicit description of the generator of Λ_k^{2k+1} for certain k.

Our results can also be used to extend somewhat results of E. Rees and E. Thomas in **[8]**. Using complex cobordism theory in place of oriented cobordism theory Rees and Thomas define, for an isolated singularity whose link has a complex normal bundle, an obstruction to smoothing it with complex structure preserved. This obstruction lies in $\pi_*(MU(n))$ for some n which depends on the dimensions involved. Note that the obstruction to orientably smoothing the same isolated singularity, as we have defined it, can be thought of as lying in $\pi_*(MSO(2n))$. It is not hard to check that the standard map $\pi_*(MU(n)) \to \pi_*(MSO(2n))$ carries one obstruction element to the other. Close examination of the generators of $\pi_{4n+1}(MU(n))$ given in **[8]** then gives the following result.

6.3. PROPOSITION. *Let n be odd and suppose $n+1$ is not a power of 2. Let $i = 2^s$ be the lowest power of 2 which occurs in the dyadic expansion of $n+1$. Then the obstruction to smoothing (with complex normal bundle) the complex cone on the projection of the Segre embedding $P_i \times P_{n-1} \to P_{2n}$ is an element of maximal order in $\pi_{4n+1}(MU(n))$, and hence can be taken as one of the generators.*

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